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LETTER TO THE EDITOR

Path integral solution of two potentials related to the SO(2,1) dynamical algebra

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Abstract. Two classes of generalized Coulomb potentials related to the SO(2,1) dynamical algebra are rigorously solved by path integration in terms of parabolic coordinates.

1. Introduction

In this paper I want to discuss two classes of potentials related to the SO(2,1) dynamical symmetry. They are

(i) $(r = \sqrt{x^2 + y^2 + z^2})$

$$\begin{aligned} V_1(x,y,z) &= -\frac{\alpha}{r} + \frac{\hbar^2}{2m}\left(\frac{b_1 + b_2}{x^2 + y^2} + \frac{(b_1 - b_2)z}{(x^2 + y^2)r}\right) \\ &\quad + \frac{m}{4}(\omega_1^2 + \omega_2^2) - \frac{m}{4}(\omega_1^2 - \omega_2^2)\frac{z}{r} \\ &= \frac{1}{\xi^2 + \eta^2}\left(\frac{m}{2}\omega_1^2\xi^2 + \frac{\hbar^2 b_1}{m\xi^2} + \frac{m}{2}\omega_2^2\eta^2 + \frac{\hbar^2 b_2}{m\eta^2} - 2\alpha\right) = V_1(\xi,\eta) \qquad (1) \end{aligned}$$

with parabolic coordinates $x = \xi\eta\cos\phi$, $y = \xi\eta\sin\phi$ and $z = \frac{1}{2}(\eta^2 - \xi^2)$, where $0 \leqslant \xi, \eta < \infty$, $0 \leqslant \phi \leqslant 2\pi$.

(ii) $(\rho = \sqrt{x^2 + y^2})$

$$\begin{aligned} V_2(x,y,z) &= -\frac{\alpha}{\rho} + \frac{\hbar^2}{2m}\left(\frac{\beta - \frac{1}{4}}{y^2} - \frac{\gamma x}{y^2\rho} + \frac{\delta^2 - \frac{1}{4}}{z^2}\right) + \frac{m}{16}(\omega_1^2 + \omega_2^2) \\ &\quad + \frac{m}{16}(\omega_1^2 - \omega_2^2)\frac{x}{\rho} + \frac{m}{2}\omega_3^2 z^2 \\ &= \frac{1}{4(u^2 + v^2)}\left[-4\alpha + \frac{\hbar^2}{2m}\left(\frac{\beta + \gamma - \frac{1}{4}}{u^2} + \frac{\beta - \gamma - \frac{1}{4}}{v^2}\right)\right. \\ &\quad \left. + \frac{m}{2}(\omega_1^2 u^2 + \omega_2^2 v^2)\right] + \frac{m}{2}\omega_3^2 z^2 + \frac{\hbar^2}{2m}\frac{\delta^2 - \frac{1}{4}}{z^2} = V_2(u,v,z) \qquad (2) \end{aligned}$$

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with $x = u^2 - v^2$, $y = 2uv$, $-\infty < u, v < \infty$, $z > 0$. Note that due to the strong singularity at $y = 0$, the regions $(-\infty < y < 0)$ and $(0 < y < \infty)$ are decoupled such that it is sufficient to consider only the domain $0 < y, z < \infty$, $x \in \mathbb{R}$, respectively, $0 < u, v, z < \infty$. The corresponding classical Lagrangians for the two potentials in two-and three-dimensional parabolic coordinates, respectively, have the form

$$\begin{aligned}\mathcal{L}_1(\xi,\eta,\phi,\dot\xi,\dot\eta,\dot\phi) &= \tfrac{1}{2}m[(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)+\xi^2\eta^2\dot\phi^2] - V_1(\xi,\eta)\\ \mathcal{L}_2(u,v,z,\dot u,\dot v,\dot z) &= \tfrac{1}{2}m[4(u^2+v^2)(\dot u^2+\dot v^2)+\dot z^2] - V_2(u,v,z).\end{aligned} \qquad (3)$$

V_1 can be seen as a highly distorted spherical Coulomb field with an additional double ring well, and V_2 as a similar highly distorted cylindrical Coulomb field, respectively. These two highly singular non-isotropic potentials have recently been discussed by Boschi-Filho *et al* [1] by the algebraic method exploiting the underlying SO(2,1) Lie algebra. The potentials of [1] generalize the similar but easier anisotropic potentials as discussed by Carpio-Bernido and Bernido [2], Boschi-Filho and Vaidya [3] (algebraic methods) and Chetouani *et al* [4], Carpio-Bernido and Bernido [5] Carpio-Bernido *et al* [6], Carpio-Bernido [7] and Grosche [8] (path integral methods). Both potentials look intractable in Cartesian coordinates as well as in polar coordinates. However, if rewritten in terms of two- and three-dimensional parabolic coordinates, the 'radial-harmonic-oscillator' structure is clearly revealed and therefore parabolic coordinates are suitable for a path integral treatment. Parabolic coordinates have also been used in the path integral discussion of the Coulomb and related potentials (Chetouani and Hammann [9], Grosche [8]) and the Kaluza–Klein monopole problem [10]. Note that both potentials (1,2) do not admit a separation in polar coordinates.

In order to set up the path integral formulation, I follow the canonical approach [11, 12] and I use a product form formulation as described in [13]. Here we have for the quantum Hamiltonian

$$H = -\frac{\hbar^2}{2m}\Delta_{\mathrm{LB}} + V(q) = \frac{1}{2m}h^{ac}p_a p_b h^{bc} + V(q) + \Delta V(q) \qquad (4)$$

where it is assumed that a decomposition $g_{ab} = h_{ac}h_{cb}$ of the metric tensor exists, Δ_{LB} is the Laplace–Beltrami operator and $p_a = -\mathrm{i}\hbar(\partial_a + \Gamma_a/2)$ are the canonical momenta ($\Gamma_a = \partial \ln\sqrt{g}$). ΔV is a well defined quantum potential

$$\begin{aligned}\Delta V(q) = \frac{\hbar^2}{8m}[&g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_a)_{,b} + g^{ab}{}_{,ab} + 2h^{ac}h^{bc}{}_{,ab}\\ &- h^{ac}{}_{,a}h^{bc}{}_{,b} - h^{ac}{}_{,b}h^{bc}{}_{,a}]\end{aligned} \qquad (5)$$

arising from the specific ordering prescription in the quantum Hamiltonian (4). For the path integral this yields

$$\begin{aligned}K(q'',q';T) &= \int \sqrt{g}\mathcal{D}q(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}h_{ac}h_{cb}\dot q^a\dot q^b - V(q) - \Delta V(q)\right]\mathrm{d}t\right\}\\ &= \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{ND/2}\prod_{j=1}^{N-1}\int \mathrm{d}q_{(j)}\sqrt{g(q_{(j)})}\\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}h_{bc}(q_{(j)})h_{ac}(q_{(j-1)})\Delta q^a_{(j)}\Delta q^b_{(j)}\right.\right.\\ &\quad\left.\left. - \epsilon V(q_{(j)}) - \epsilon\Delta V(q_{(j)})\right]\right\}.\end{aligned} \qquad (6)$$

Here we put $\Delta q_{(j)} = q_{(j)} - q_{(j-1)}$ for $q_{(j)} = q(t' + j\epsilon)$ ($\epsilon = (t'' - t')/N = T/N$, $j = 1, \ldots, N$ in the limit $N \to \infty$), and D is the spatial dimension.

I will discuss now these two potentials in the path integral formulation by means of two- and three-dimensional parabolic coordinates.

2. The potential $V_1(\xi, \eta)$

In the parabolic coordinates for the potential V_1 we have $\Delta V_1(\xi,\eta) = -\hbar^2/8m\xi^2\eta^2$, and consequently for the path integral

$$\begin{aligned}
K(\xi'', \xi', \eta'', \eta', \phi'', \phi'; T)
&= \int \mathcal{D}\xi(t) \int \mathcal{D}\eta(t) (\xi^2 + \eta^2)\xi\eta \\
&\quad \times \int \mathcal{D}\phi(t) \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[\mathcal{L}_1(\xi, \eta, \phi, \dot{\xi}, \dot{\eta}, \dot{\phi}) + \frac{\hbar^2}{8m\xi^2\eta^2} \right] \mathrm{d}t \right\} \\
&= \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu(\phi'' - \phi')} K_\nu(\xi'', \xi', \eta'', \eta'; T)
\end{aligned} \tag{7}$$

where I have separated the ϕ-dependence according to [14] with $K_\nu(T)$ given by

$$\begin{aligned}
K_\nu(\xi'', \xi', \eta'', \eta', \phi'', \phi'; T) &= (\xi'\xi''\eta'\eta'')^{-1/2} \int \mathcal{D}\xi(t) \int \mathcal{D}\eta(t) (\xi^2 + \eta^2) \\
&\quad \times \exp\left(\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left\{ \frac{m}{2}(\xi^2 + \eta^2)(\dot{\xi}^2 + \dot{\eta}^2) - \frac{1}{\xi^2 + \eta^2} \left[\frac{m}{2}(\omega_1^2 \xi^2 + \omega_2^2 \eta^2) \right.\right.\right. \\
&\quad \left.\left.\left. + \frac{\hbar^2}{2m} \left(\frac{2b_1 + \nu^2 - \frac{1}{4}}{\xi^2} + \frac{2b_2 + \nu^2 - \frac{1}{4}}{\eta^2} \right) - 2\alpha \right] \right\} \mathrm{d}t \right).
\end{aligned} \tag{8}$$

Now a time transformation [15, 16] is performed with its continuous and lattice implementation, respectively

$$s(t) = \int_{t'}^{t} \frac{\mathrm{d}\sigma}{\xi^2 + \eta^2} \qquad s'' = s(t'') \qquad \epsilon = \delta(\widehat{\xi_{(j)}^2 + \eta_{(j)}^2}) \tag{9}$$

where $\widehat{f_{(j)}^2} \equiv f(q_{(j)}) f(q_{(j-1)})$ for some function of the coordinates. Of course, we identify $\xi(t'') = \xi(s(t'')) = \xi(s'') \equiv \xi''$, etc. This gives the transformation formulae

$$\begin{aligned}
K_\nu(\xi'', \xi', \eta'', \eta'; T) &= \frac{1}{2\pi\mathrm{i}\hbar} \int_{-\infty}^{\infty} \mathrm{d}E\, \mathrm{e}^{-\mathrm{i}ET/\hbar} G_\nu(\xi'', \xi', \eta'', \eta'; E) \\
G_\nu(\xi'', \xi', \eta'', \eta'; E) &= \mathrm{i} \int_0^\infty \mathrm{d}s''\, \mathrm{e}^{2\mathrm{i}\alpha s''/\hbar} \tilde{K}_\nu(\xi'', \xi', \eta'', \eta'; s'')
\end{aligned} \tag{10}$$

with the transformed kernel $\tilde{K}_\nu(s'')$ given by

$$\tilde{K}_\nu(\xi'', \xi', \eta'', \eta'; s'') = \tilde{K}_\xi(\xi'', \xi'; s'') \times \tilde{K}_\eta(\eta'', \eta'; s'') \tag{11}$$

thus decoupling into two kernels $\tilde{K}_\xi(s'')$ and $\tilde{K}_\eta(s'')$ which in turn are given by

$$\tilde{K}_\xi(\xi'',\xi';s'') = \frac{m\Omega_1}{\mathrm{i}\hbar\sin\Omega_1 s''}\exp\left[-\frac{m\Omega_1}{2\mathrm{i}\hbar}(\xi'^2+\xi''^2)\cot\Omega_1 s''\right] I_{\lambda_1}\left(\frac{m\Omega_1\xi'\xi''}{\mathrm{i}\hbar\sin\Omega_1 s''}\right) \tag{12}$$

and $\tilde{K}_\eta(\eta'',\eta';s'')$ with all indices replaced by $1\to 2$, ($\Omega_{1/2}=\sqrt{\omega_{1/2}^2-2E/m}$, $\lambda_{1/2}=\sqrt{2b_{1/2}+\nu^2}$). Here the well known Peak–Inomata formula [17, 18] for radial path integrals has been applied

$$\begin{aligned}
\int \mathcal{D}r(t)\exp&\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot r^2-\frac{m}{2}\omega^2 r^2-\frac{\hbar^2}{2m}\frac{\lambda^2-\frac14}{r^2}\right]\mathrm{d}t\right\}\\
&\equiv \int \mu_\lambda[r^2]\mathcal{D}r(t)\exp\left[\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}(\dot r^2-\omega^2 r^2)\,\mathrm{d}t\right]\\
&=\frac{m\omega\sqrt{r'r''}}{\mathrm{i}\hbar\sin\omega T}\exp\left[-\frac{m\omega}{2\mathrm{i}\hbar}(r'^2+r''^2)\cot\omega T\right] I_\lambda\left(\frac{m\omega r'r''}{\mathrm{i}\hbar\sin\omega T}\right)
\end{aligned} \tag{13}$$

with the functional weight $\mu_\lambda[r^2]$ as defined in [12, 19]

$$\begin{aligned}
\mu_\lambda[r^2] &= \lim_{N\to\infty}\prod_{j=1}^{N}\mu_\lambda[r_{(j-1)}r_{(j)}]\\
&:= \lim_{N\to\infty}\prod_{j=1}^{N}\left(\frac{2\pi m r_{(j-1)}r_{(j)}}{\mathrm{i}\epsilon\hbar}\right)^{1/2}\exp\left(\frac{m r_{(j-1)}r_{(j)}}{\mathrm{i}\epsilon\hbar}\right) I_\lambda\left(\frac{m r_{(j-1)}r_{(j)}}{\mathrm{i}\epsilon\hbar}\right)
\end{aligned} \tag{14}$$

in order to guarantee a well defined short-time kernel. Let us remark that, according to Fischer *et al* [20], this functional weight formulation $\mu_\lambda[r^2]$ is completely equivalent to the path integral formulation of [17] with angular dependence $\propto \hbar^2(\lambda^2-\frac14)/2mr^2$ in the action. Putting everything together I obtain an integral representation for the Green function $G(E)$

$$\begin{aligned}
G(\xi'',\xi',\eta'',\eta',\phi'',\phi';E) &= \frac{\mathrm{i}}{2\pi}\left(\frac{m}{\mathrm{i}\hbar}\right)^2\sum_{\nu=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\nu(\phi''-\phi')}\Omega_1\Omega_2\\
&\times\int_0^\infty\frac{\mathrm{d}s''\,\mathrm{e}^{2\mathrm{i}\alpha s''/\hbar}}{\sin\Omega_1 s''\sin\Omega_2 s''} I_{\lambda_1}\left(\frac{m\Omega_1\xi'\xi''}{\mathrm{i}\hbar\sin\Omega_1 s''}\right) I_{\lambda_2}\left(\frac{m\Omega_2\eta'\eta''}{\mathrm{i}\hbar\sin\Omega_2 s''}\right)\\
&\times\exp\left\{-\frac{m}{2\mathrm{i}\hbar}[\Omega_1(\xi'^2+\xi''^2)\cot\Omega_1 s''+\Omega_2(\eta'^2+\eta''^2)\cot\Omega_2 s'']\right\}.
\end{aligned} \tag{15}$$

Note that the 'addition theorem' for Bessel functions as used in [6, 8, 10, 15] cannot be applied due to $\Omega_1\neq\Omega_2$ in general. This also shows the non-separability in polar coordinates.

To determine the wavefunctions and the energy spectrum, respectively, I make use of the Hille–Hardy formula [21, p 1038]

$$\frac{t^{-\lambda/2}}{1-t}\exp\left(-\frac{x+y}{2}\frac{1+t}{1-t}\right)I_\lambda\left(\frac{2\sqrt{xyt}}{1-t}\right) = \sum_{n=0}^{\infty}\frac{t^n\,n!}{\Gamma(n+\lambda+1)}(xy)^{\lambda/2}L_n^{(\lambda)}(x)\,L_n^{(\lambda)}(y)\mathrm{e}^{-(x+y)/2}. \tag{16}$$

After performing the s'' integration this yields the quantization condition

$$\Omega_1(2n_1+\lambda_1+1)+\Omega_2(2n_2+\lambda_2+1)-2\alpha/\hbar=0. \tag{17}$$

Therefore we get for the bound-state contribution of the Green function

$$G_{\mathrm{bound}}(\xi'',\xi',\eta'',\eta',\phi'',\phi';E) = \sum_{n_1,n_2=0}^{\infty}\sum_{\nu=-\infty}^{\infty}\frac{1}{E-E_{n_1,n_2}}\Psi_{n_1,n_2,\nu}(\xi',\eta',\phi')\Psi^*_{n_1,n_2,\nu}(\xi'',\eta'',\phi'') \tag{18}$$

with the wavefunctions

$$\Psi_{n_1,n_2,\nu}(\xi,\eta,\phi) = \frac{\mathrm{e}^{\mathrm{i}\nu\phi}}{\sqrt{2\pi}}\left[\left(\frac{m}{\hbar}\right)^3\frac{(2\Omega_1\Omega_2)^2}{A_1\Omega_2+A_2\Omega_1}\frac{n_1!n_2!}{\Gamma(n_1+\lambda_1+1)\Gamma(n_2+\lambda_2+1)}\right]^{1/2} \times\left(\frac{m\Omega_1}{\hbar}\xi^2\right)^{\lambda_1/2}\left(\frac{m\Omega_2}{\hbar}\eta^2\right)^{\lambda_2/2}L_{n_1}^{(\lambda_1)}\left(\frac{m\Omega_1}{\hbar}\xi^2\right)L_{n_2}^{(\lambda_2)}\left(\frac{m\Omega_2}{\hbar}\eta^2\right) \times\exp\left[-\frac{m}{2\hbar}(\Omega_1\xi^2+\Omega_2\eta^2)\right] \tag{19}$$

and the energy spectrum has the form

$$E_{n_1,n_2} = \frac{m/2}{(A_1^2-A_2^2)^2}\left[(A_1^2-A_2^2)(A_1^2\omega_1^2-A_2^2\omega_2^2)-\frac{4\alpha^2}{\hbar^2}(A_1^2+A_2^2) +\frac{4\alpha}{\hbar}A_1A_2\sqrt{(A_1^2-A_2^2)(\omega_1^2-\omega_2^2)+\frac{4\alpha^2}{\hbar^2}}\right]. \tag{20}$$

These results are equivalent to those in [1]. Here denote $A_{1/2}=2n_{1/2}+\lambda_{1/2}+1$, with

$$\Omega_{1/2}=\frac{1}{|A_1^2-A_2^2|}\left|A_{2/1}\sqrt{(A_1^2-A_2^2)(\omega_1^2-\omega_2^2)+\frac{4\alpha^2}{\hbar^2}}-\frac{2\alpha}{\hbar}A_{1/2}\right| \tag{21}$$

and all quantities are valid for $A_1\neq A_2$. For $A_1=A_2=A$ $(\omega_1\neq\omega_2)$ one obtains for the energy spectrum

$$E_A=\frac{m}{4}(\omega_1^2+\omega_2^2)-\frac{m\alpha^2}{2\hbar^2A^2}-\frac{m\hbar^2A^2}{32\alpha^2}(\omega_1^2-\omega_2^2). \tag{22}$$

For the special case $\omega_1 = \omega_2 = \omega$ $(A_1 \neq A_2)$ we get

$$E_{n_1,n_2} = \frac{m}{2}\omega^2 - \frac{m\alpha^2}{2\hbar^2(\frac{1}{2}(A_1 + A_2))^2} \tag{23}$$

and this case fixes the sign in the square-root expressions. In the same limit:

$$m\Omega_{1/2}/\hbar \to 2m|\alpha|/[\hbar^2(A_1 + A_2)]$$

and

$$(m/\hbar)^3(2\Omega_1\Omega_2)^2/(A_1\Omega_2 + A_2\Omega_1) \to 2/[a^3(\tfrac{1}{2}(A_1 + A_2))^4]$$

$(\omega_1 \to \omega_2)$, $(a = \hbar^2/m|\alpha|$—the Bohr radius) and all three quantities are the correct expressions for the Hartmann potential (compare, e.g., [1, 5–8]). The correct normalization of the bound-state wavefunctions is checked by the property of the Coulomb wavefunctions. (Note the importance of the absolute values in $\Omega_{1/2}$ for the correct evaluation of the residua of the Green function at the poles of the energy spectrum.)

3. The potential $V_2(u, v, z)$

As for V_1 we formulate the path integral for two-dimensional parabolic coordinates. Here we have $\Delta V_2 = 0$. This gives

$$\begin{aligned} K(u'', u', v'', v', z'', z'; T) &= \int \mathcal{D}u(t) \int \mathcal{D}v(t) 4(u^2 + v^2) \\ &\quad \times \int \mathcal{D}z(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \mathcal{L}_2(u, v, z, \dot{u}, \dot{v}, \dot{z})\,\mathrm{d}t\right] \\ &= K_{uv}(u'', u', v'', v'; T) \times K_z(z'', z'; T) \end{aligned} \tag{24}$$

where the z-dependence separates immediately with $K_z(T)$ given by

$$K_z(z'', z'; T) = \frac{m\omega_3\sqrt{z'z''}}{\mathrm{i}\hbar \sin\omega_3 T} \exp\left[-\frac{m\omega_3}{2\mathrm{i}\hbar}(z'^2 + z''^2)\cot\omega_3 T\right] I_\delta\left(\frac{m\omega_3 z'z''}{\mathrm{i}\hbar\sin\omega_3 T}\right). \tag{25}$$

The remaining (u, v) path integrations have the form

$$\begin{aligned} K_{uv}(u'', u', v'', v'; T) &= \int \mathcal{D}u(t) \int \mathcal{D}v(t) 4(u^2 + v^2) \\ &\quad \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left[2m(u^2 + v^2)(\dot{u}^2 + \dot{v}^2) - \frac{1}{4(u^2 + v^2)}\right.\right. \\ &\quad \left.\left. \times \left(-4\alpha + \frac{m}{2}\omega_1^2 u^2 + \hbar^2\frac{\beta + \gamma - \frac{1}{4}}{2mu^2} + \frac{m}{2}\omega_2^2 v^2 + \hbar^2\frac{\beta - \gamma - \frac{1}{4}}{2mv^2}\right)\right]\mathrm{d}t\right\}. \end{aligned} \tag{26}$$

The appropriate time transformation now is

$$s(t) = \int_{t'}^{t} \frac{d\sigma}{4(u^2+v^2)} \qquad s'' = s(t'') \qquad \epsilon = 4\delta(\widehat{u_{(j)}^2 + v_{(j)}^2}). \quad (27)$$

We repeat the steps of equations (10) and (11) (just replace $\nu \to uv$, $\xi \to u$, $\eta \to v$ and $\alpha \to 2\alpha$) and we arrive at

$$\tilde{K}_{uv}(u'', u', v'', v'; s'') = \tilde{K}_u(u'', u'; s'') \times \tilde{K}_v(v'', v'; s''). \quad (28)$$

Therefore again I have achieved a decoupling of the u and v path integrations with the kernels $K_u(s'')$ and $K_v(s'')$ given by†

$$\tilde{K}_u(u'', u'; s'') = \frac{m\Omega_1\sqrt{u'u''}}{i\hbar \sin\Omega_1 s''} \exp\left[-\frac{m\Omega_1}{2i\hbar}(u'^2 + u''^2)\cot\Omega_1 s''\right] \times I_{\lambda_1}\left(\frac{m\Omega_1 u'u''}{i\hbar\sin\Omega_1 s''}\right) \quad (29)$$

and $\tilde{K}_v(v'', v'; s'')$ with all indices replaced by $1 \to 2$. The corresponding Green function $G_{uv}(E)$ is constructed in the same way as for $V_1(\xi, \eta)$. Expanding the kernels by means of the Hille–Hardy formula, we obtain, after performing the s'' integration, the quantization condition $\Omega_1(2n_1+\lambda_1+1)+\Omega_2(2n_2+\lambda_2+1)-4\alpha/\hbar = 0$, and this gives, in the usual way, the energy spectrum and the bound-state wavefunctions, respectively,

$$E_{n_1,n_2} = \frac{m/8}{(A_1^2 - A_2^2)^2}\left[(A_1^2 - A_2^2)(A_1^2\omega_1^2 - A_2^2\omega_2^2) - \frac{16\alpha^2}{\hbar^2}(A_1^2 + A_2^2) + \frac{8\alpha}{\hbar}A_1A_2\sqrt{(A_1^2 - A_2^2)(\omega_1^2 - \omega_2^2) + \frac{16\alpha^2}{\hbar^2}}\right] + \omega_3(2n_3 + \delta + 1) \quad (30)$$

$$\begin{aligned}\Psi_{n_1,n_2,n_3}(u, v, z) &= \left[2\sqrt{\frac{m\omega_3}{\hbar}}\left(\frac{m}{\hbar}\right)^3 \frac{(\Omega_1\Omega_2)^2}{A_1\Omega_2 + A_2\Omega_1} \cdot \frac{n_1!n_2!}{\Gamma(n_1+\lambda_1+1)\Gamma(n_2+\lambda_2+1)}\right]^{1/2} \\ &\times \sqrt{uvz}\left(\frac{m\Omega_1}{\hbar}u^2\right)^{\lambda_1/2} L_{n_1}^{(\lambda_1)}\left(\frac{m\Omega_1}{\hbar}u^2\right)\left(\frac{m\Omega_2}{\hbar}v^2\right)^{\lambda_2/2} \\ &\times L_{n_2}^{(\lambda_2)}\left(\frac{m\Omega_2}{\hbar}v^2\right)\left(\frac{m\omega_3}{\hbar}z^2\right)^{\delta/2} L_{n_3}^{(\delta)}\left(\frac{m\omega_3}{\hbar}z^2\right) \\ &\times \exp\left[-\frac{m}{2\hbar}(\Omega_1u^2 + \Omega_2v^2 + \omega_3z^2)\right]. \end{aligned} \quad (31)$$

Here $A_{1/2}$ and $\Omega_{1/2}$ are similarly given as in equation (21) (replace $\alpha \to 2\alpha$ and note the remark above), and all considerations from the previous section can be made analogously. n_3 denotes the quantum number arising from expanding equation (25) by means of the Hille–Hardy formula.

† $\lambda_{1/2} = \sqrt{\beta \pm \gamma}$, $\Omega_{1/2} = \sqrt{\omega_{1/2}^2 - 8E/m}$, these quantities must not be confused with those from section 2.

4. Discussion

In this letter I have used the path integration technique to solve two highly asymmetrical two- and three-dimensional Coulomb-like potentials, which are generated by a SO(2,1) dynamical algebra. The problems in question were only separable in two- and three-dimensional parabolic coordinates, respectively. In both cases a time transformation was needed to reveal the underlying 'radial harmonic-oscillator' structure of the two potentials. The bound-state wavefunctions and the energy spectrum were explicitly evaluated and compared with respect to limiting cases.

Let us stress that the approach by the Kustaanheimo–Stiefel transformation (compare with [6, 8, 15]) by which the path integral for the hydrogen atom was solved by Duru and Kleinert [15] can also be used to evaluate the path integral for the potential $V_1(x, y, z)$. However, the calculation presented here is much simpler than by the Kustaanheimo–Stiefel transformation [22] because

(i) the use of the Kustaanheimo–Stiefel transformation requires the introduction of a fourth auxiliary variable x_4 which complicates the path integral calculation considerably and makes it somewhat ambiguous; and

(ii) the transformation from three-dimensional Cartesian coordinates to three-dimensional parabolic coordinates is, in fact, closely related to the Kustaanheimo–Stiefel transformation such that after integrating out the auxiliary variable x_4 equation (15) is recovered.

Note that the transformation from two-dimensional Cartesian coordinates to two-dimensional parabolic coordinates in the case of $V_2(x, y, z)$ is, in fact, a two-dimensional Kustaanheimo–Stiefel transformation [15, 23].

Therefore I have added two further instructive examples to the list of exactly solvable path integrals.

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